



On the conditions under which instantaneous mass-transfer mappings reproduce conservative limits

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Abstract

This paper analyzes a specific instantaneous mass-transfer mapping in which a discrete mass element is relocated from the donor to the accretor and assigned the accretor's velocity. We show that this mapping changes orbital angular momentum at each transfer event by an amount proportional to $\delta m (M_1^2 - M_2^2) / M_2^2$, even when total linear momentum is exactly conserved. We derive the cumulative drift in closed form: after transferring a fraction f of the donor mass, $L_f / L_0 = (1 - f)(1 + f q_0)$, where q_0 is the initial mass ratio. A consequence of this result is that the unclosed mapping leaves the orbital separation unchanged ($a_f / a_0 = 1$), because the angular momentum drift and reduced-mass change cancel exactly—the orbit is frozen while mass is redistributed. In illustrative experiments transferring 15% of the donor mass, the resulting orbital deviation from the conservative prediction reaches 1–22% depending on mass ratio, and scales $\sim$ linearly with the total transferred mass, in quantitative agreement with the analytical formula. The drift is a mathematical property of the mapping definition, not a discretization artifact, and, for this mapping, does not vanish as $\delta m \rightarrow 0$. We derive and implement a closure condition—a uniform rescaling of tangential velocities after each transfer event—that restores angular momentum conservation and recovers the continuous conservative limit. With this closure active, the isolated two-body simulations achieve $10^{-4}\%$ agreement with analytical predictions for all mass ratios tested. The closure condition may serve as a diagnostic or correction in numerical simulations that represent mass transfer through discrete position–velocity updates, such as interacting binaries embedded in dynamical N -body simulations. As a proof of concept, we also test the closure in hierarchical three-body systems and find that it suppresses spurious transfer-induced angular-momentum jumps in the inner binary while preserving the expected qualitative orbital response under both weak and stronger external perturbations.

Keywords

Binary mass transfer, Orbital angular momentum conservation, Interacting binary stars, N -body stellar dynamics, Conservative mass transfer, Operator splitting methods, Roche lobe overflow dynamics, Stellar binary evolution, Few-body gravitational dynamics, Computational astrophysics

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1 Introduction

The Milky Way Galaxy contains many binary star systems. In fact, surveys suggest that roughly half of all solar-type stars have at least one stellar companion (1). Mass transfer can occur between them when the orbital separation of the two stars becomes sufficiently small, due to the donor's evolutionary expansion or due to any form of angular momentum loss (2, 3). This process can dramatically alter both stellar structure and orbital parameters.

The theoretical framework for conservative mass transfer—in which total system mass and orbital angular momentum are both conserved—was established by Paczyński (4) and has since become a cornerstone of binary evolution theory (5). Under this framework, the orbital response to mass transfer is fully determined by the energy conservation laws. This analytical result is widely used to predict the long-term fate of interacting binaries, from stable Roche lobe overflow to common envelope evolution (6,7).

In practice, different codes implement mass transfer in different ways. Analytic binary evolution and population synthesis codes such as BSE (9) and COSMIC (10) evolve orbital parameters directly using formulas derived from conservation laws; the premise we discuss here does not arise in these codes. Similarly, detailed stellar evolution codes such as MESA (8) resolve mass transfer through differential equations for orbital elements, and hydrodynamic codes (e.g., SPH) resolve the transfer stream directly, so that angular

momentum transport emerges naturally from the equations of motion.

However, there exist contexts in which mass transfer must be represented as a discrete update to the positions and velocities of individual bodies rather than to orbital elements. This situation arises, for example, in dynamical N -body simulations of dense stellar environments, where binaries interact gravitationally with passing stars and where analytic orbital-element prescriptions cannot always be applied. In such environments, a natural approach is to instantaneously relocate a mass element δm from the donor to the accretor in position-velocity space.

This paper analyzes one such instantaneous relocation mapping: a specific operator in which the transferred mass element is removed from the donor and deposited on the accretor with the accretor's velocity. Other prescriptions are possible—for example, assigning the donor's velocity or an intermediate stream velocity—and these would produce different angular momentum behavior.

Our goal is not to critique existing binary evolution codes, but to characterize the angular momentum properties of this particular mapping and to derive a closure condition that restores consistency with the conservative analytical limit. The analysis reveals that linear momentum conservation ($P=0$) during an instantaneous transfer step does not guarantee angular momentum conservation ($\dot{L}=0$). This distinction does not arise in the continuous analytical framework where most theory is

developed, but becomes relevant whenever mass transfer is implemented as a discrete operator on particle states.

This motivates our key research question: *under what conditions does the instantaneous relocation mapping defined here reproduce the same orbital evolution as continuous conservative mass transfer?*

To answer this question, we first analytically derive why the mapping changes orbital angular momentum, even under center-of-mass

constraints. We then derive a closure condition that restores angular momentum conservation and validates the framework numerically through a series of simulations.

2 Classical framework

2.1 Orbital angular momentum

Consider a binary system consisting of two point masses M_1 (the donor) and M_2 (the accretor) in a circular orbit with separation a . The total orbital angular momentum is given by

$$L = \frac{M_1 M_2}{M_1 + M_2} \sqrt{G(M_1 + M_2)a} = \mu \sqrt{GMa}, \quad (\text{eq1})$$

where $\mu = M_1 M_2 / (M_1 + M_2)$ is the reduced mass.

2.2 Assumptions

The conservative mass-transfer framework adopts the following standard assumptions (4,5): total mass conservation ($\dot{M}_1 + \dot{M}_2 = 0$), angular momentum conservation ($\dot{L} = 0$), and circular orbits maintained by tidal

2.3 Analytical prediction

Under these assumptions, the rate of change of

orbital separation is $\frac{\dot{a}}{a} = -2 \frac{\dot{M}_1}{M_1} \left(1 - \frac{M_1}{M_2}\right)$ (eq2).

Since $\dot{M}_1 < 0$, the orbit expands when $q < 1$ and contracts when $q > 1$. For finite mass transfer, the final separation is

$$\frac{a_f}{a_0} = \left(\frac{M_{1,0} M_{2,0}}{M_{1,f} M_{2,f}} \right)^2, \quad (\text{eq3}).$$

Equation 3 is derived under the assumption of continuous transfer with simultaneous orbital adjustment. Whether a discrete numerical implementation reproduces this result depends on how the transfer step is constructed—a question we address in the following sections.

3 Numerical framework

3.1 Design philosophy

This framework is designed to prioritize transparency and reproducibility. Rather than modeling full stellar physics, the framework isolates specific conservation properties to test

the kinematic consequences of different mass-transfer prescriptions. By explicitly enforcing conservation laws within the simulation, we can isolate the orbital response as predicted by classical theory.

3.2 Definition of the instantaneous relocation mapping

An *instantaneous relocation mapping* is a discrete operator $\tau_{\delta m}$ that acts on the system state $(M_1, M_2, r_1, r_2, v_1, v_2)$ as follows. Here r_i and v_i denote the position and velocity of each component in an inertial frame; when convenient, we specialize to the center-of-mass (COM) frame in which $M_1 r_1 + M_2 r_2 = 0$ and $M_1 v_1 + M_2 v_2 = 0$.

[1]. *Mass update:* $M_1 \rightarrow M_1 - \delta m$, $M_2 \rightarrow M_2 + \delta m$.

[2]. *Position invariance:* The mapping leaves the instantaneous positions unchanged (equivalently, the relative separation $r \equiv r_2 - r_1$ is invariant during $\tau_{\delta m}$).

[3]. *Velocity assignment:* The transferred mass element δm is removed from the donor with velocity v_1 and deposited on the accretor, where it is assigned velocity v_2 .

[4]. *No intermediate dynamics:* No gravitational evolution, torques, or orbital adjustment occurs during the mapping step itself.

This operator represents one specific computational idealization: mass is “teleported” between components and immediately assigned

the accretor’s velocity (equivalently, a perfectly inelastic capture). Other instantaneous prescriptions exist—for example, depositing mass with the donor’s velocity or with an intermediate stream velocity—and these would produce different angular momentum changes. The present analysis focuses on the accretor-comoving deposition rule because it provides a clear and analytically tractable example of an instantaneous relocation operator. The mapping conserves total mass by construction. However, as we derive below, it does not automatically conserve angular momentum.

3.3 Angular momentum change under instantaneous relocation

We now derive analytically why the mapping $\tau_{\delta m}$ changes orbital angular momentum even when total linear momentum is constrained to zero.

The orbital angular momentum of the binary is $L = M_1 r_1 \times v_1 + M_2 r_2 \times v_2$ (eq4). Before the mapping, the mass element δm contributes to L through its association with the donor:

$L_{\delta m}^{[before]} = \delta m r_1 \times v_1$ (eq5). After instantaneous relocation, the same mass element is located at

r_2 with velocity v_2 , as $L_{\delta m}^{[after]} = \delta m r_2 \times v_2$ (eq6).

The change in angular momentum due to a single transfer event is therefore $\Delta L = \delta m (r_2 \times v_2 - r_1 \times v_1)$ (eq7). In the center-of-

mass frame, $r_2 = -\frac{M_1}{M_2} r_1$ and $v_2 = -\frac{M_1}{M_2} v_1$ (for circular orbits). Substituting, we obtain,

$$\Delta L = \delta m \left[\left(-\frac{M_1}{M_2} \right)^2 - 1 \right] r_1 \times v_1 = \delta m \left(\frac{M_1^2 - M_2^2}{M_2^2} \right) r_1 \times v_1, \quad (\text{eq8})$$

This result has several important implications,
[1]. $\Delta L \neq 0$ unless $M_1 = M_2$ (equal-mass case).

[2]. The sign of ΔL depends on whether $M_1 > M_2$ or $M_1 < M_2$: angular momentum increases when the donor is more massive and decreases when the accretor is more massive.

[3]. The drift is proportional to δm but does *not* vanish as $\delta m \rightarrow 0$ in a fixed-total-mass sense: more events are required to transfer the same total mass, and the cumulative drift scales with total transferred mass.

[4]. The center-of-mass constraint ($P=0$) is satisfied throughout; the angular momentum change arises from redistribution, not from linear momentum violation. Equation 8 demonstrates that the angular momentum drift is a mathematical consequence of the mapping definition, not a numerical artifact. This result applies specifically to the accretor-comoving

deposition rule defined in section 3.2.

3.4 Cumulative drift in the continuous limit

The per-step result (eq8) can be integrated to obtain a closed-form expression for the cumulative angular momentum drift. Dividing Equation 8 by L and using $L = M_1 M_{\text{tot}} l_1 / M_2$ (where $M_{\text{tot}} = M_1 + M_2$ and $l_1 = |r_1 \times v_1|$), the fractional change per step simplifies to

$$\frac{\Delta L}{L} = \delta m \left(\frac{1}{M_2} - \frac{1}{M_1} \right), \quad (\text{eq9}).$$

In the continuous limit, with $M_1(m) = M_{1,0} - m$ and $M_2(m) = M_{2,0} + m$ after transferring cumulative mass m , we integrate

$$\ln \frac{L_f}{L_0} = \int_0^{M_{1,0}} \left[\frac{1}{M_{2,0} + m} - \frac{1}{M_{1,0} - m} \right] dm, \quad (\text{eq10}).$$

Both integrals are elementary. Evaluating them gives

$$\frac{L_f}{L_0} = (1-f)(1+f q_0), \quad (\text{eq11})$$

where $f = \Delta M / M_{1,0}$ is the fractional donor mass transferred and $q_0 = M_{1,0} / M_{2,0}$ is the initial mass ratio. This result has a notable consequence for the orbital separation. For a circular orbit, $a \propto L^2 / \mu^2$ (with M_{tot} conserved),

hence, $\frac{a_f}{a_0} = \left(\frac{L_f}{L_0} \right)^2 \left(\frac{\mu_0}{\mu_f} \right)^2$, (eq12). Since the final

masses are $M_{1,f} = M_{1,0}(1-f)$ and $M_{2,f} = M_{2,0}(1+f q_0)$ regardless of whether closure is applied, we have $\mu_0 / \mu_f = 1 / [(1-f)(1+f q_0)]$. Substituting Equation (eq11), we obtain

$$\frac{a_f}{a_0} \bigg|_{unclosed} = \frac{[(1-f)(1+f q_0)]^2}{[(1-f)(1+f q_0)]^2} = 1, \quad (\text{eq13}).$$

That is, the unclosed mapping produces no net change in orbital separation in the continuous limit. The angular momentum drift and the reduced-mass change cancel exactly, leaving the orbit frozen at its initial value. Physically, the mapping relocates mass without supplying the orbital response that, in continuous transfer, would be mediated by internal torques and dynamical adjustment. The closure condition (Section 3.6) supplies precisely this missing response.

The fractional angular momentum drift itself is $\frac{L_f - L_0}{L_0} = f(q_0 - 1) - f^2 q_0$, (eq14), which is linear in f for small transferred fractions ($\approx f(q_0 - 1)$ when $f \ll 1$). For the equal-mass case $q_0 = 1$, the

linear term vanishes and the cumulative drift begins at second order: $(L_f - L_0)/L_0 = -f^2$. Setting Equation (eq14) to zero gives a critical mass ratio as $q_{0,crit} = \frac{1}{1-f}$, (eq15), at which the cumulative drift changes sign: for $q_0 < q_{0,crit}$ the unclosed mapping decreases L , while for $q_0 > q_{0,crit}$ it increases L . At $f = 0.15$, $q_{0,crit} \approx 1.18$; the sign boundary shifts to higher mass ratios as more mass is transferred.

The orbital deviation of the unclosed mapping relative to the conservative prediction is

$$\text{Error} = \frac{|1 - (a_f/a_0)_{cons}|}{(a_f/a_0)_{cons}} \times 100\%, \quad (\text{eq16}), \quad \text{since } (a_f/a_0)_{unclosed} = 1 \text{ exactly. These predictions are verified numerically in Section 4.}$$

3.5 Dynamical integration

Table 1. Simulation parameters for validation tests.

Parameter	Value
Timestep	$10^{-4} P_{orb}$ (10,000 steps per orbit)
Number of orbits	50
Transfer events	500 (10 per orbit)
Mass per event	$\delta m = 0.0003 M_1$ (total: 15% of initial M_1)
Initial separation	$a_0 = 1 \text{ AU}$ ($1.496 \times 10^{11} \text{ m}$)
Accretor mass	$M_2 = 1 M_\odot$ ($1.989 \times 10^{30} \text{ kg}$)
P_{scale} (Table 2)	$\sqrt{G M_{tot} / a_0}$

The binary is modeled as two point masses interacting through Newtonian gravity. The equations of motion are integrated using the velocity Verlet algorithm (12). In the absence of mass transfer, the integrator conserves energy and angular momentum to machine precision ($<10^{-14}$) over hundreds of orbital periods. Table 1 summarizes the simulation parameters used in all validation tests. One astronomical unit (AU) is defined as 1.496×10^{11} m. One solar mass ($M_{\odot}$) is defined as 1.989×10^{30} kg.

3.6 Angular momentum closure condition

The derivation in Section 3.3 shows that instantaneous relocation changes L at each event. To enforce physical equivalence with continuous conservative transfer—where mass flow and orbital adjustment occur together, preserving L —we impose an explicit *closure condition* after each mapping step.

The closure represents the unresolved internal dynamics: in a real transfer process, torques and orbital adjustment would redistribute angular momentum on dynamical timescales. The instantaneous mapping cannot resolve these processes; the closure explicitly supplies their net effect.

3.6.1 Implementation

Working in the center-of-mass frame (where $P=0$ by construction), we decompose each body's velocity into radial and tangential components as described by equation 17, $\mathbf{v}_i = \mathbf{v}_{i,rad} + \mathbf{v}_{i,\perp}$, where $\mathbf{v}_{i,rad} = (\mathbf{v}_i \cdot \hat{\mathbf{r}}_i) \hat{\mathbf{r}}_i$, (eq17).

Only the tangential components carry angular momentum. We rescale them uniformly to achieve the target angular momentum L_{target} (the pre-transfer value for conservative transfer) as,

$$\mathbf{v}_{i,\perp} \rightarrow \mathbf{v}_{i,\perp} \cdot \frac{L_{target}}{L_{current}}, \quad (\text{eq 18}).$$

This rescaling

- [1]. Preserves the center-of-mass frame ($P=0$ is maintained).
- [2]. Leaves radial velocity components unchanged (no spurious radial impulses).
- [3]. Enforces the angular momentum that continuous transfer would preserve.

For circular orbits (our test cases), the radial components are negligible and the rescaling adjusts only the orbital velocity magnitude. Because the rescaling changes kinetic energy, the closure implicitly represents the unresolved work done by internal torques during continuous

transfer; it enforces the conservative angular momentum limit, and the orbital energy adjusts accordingly. For eccentric orbits, tangential rescaling would also affect eccentricity; such cases lie outside our present scope.

3.6.2 Physical interpretation

The closure is a minimal closure enforcing angular momentum conservation. It is a natural one-parameter condition that enforces the angular momentum that continuous transfer would preserve. Other closure prescriptions are possible (e.g., adjusting only one component or applying asymmetric tangential kicks); the

uniform tangential rescaling used here is the simplest choice that preserves the center-of-mass frame. Omitting closure altogether is equivalent to assuming that mass can be relocated without any compensating orbital response—a physically distinct scenario corresponding to non-conservative or externally driven transfer.

3.7 Orbital diagnostics

To avoid phase-dependent oscillations, we compute the osculating semi-major axis as

$a_{osc} = -\frac{GM}{2\varepsilon}$, (eq19), where $\varepsilon = \frac{1}{2}v_{rel}^2 - GM/r$ is the specific orbital energy.

4 Validation tests

We validate the framework using three test cases with initial mass ratios $q_0 = M_1/M_2 = 0.5, 1.0$,

and 2.0. In all cases, $M_2 = 1 M_\odot$, $a_0 = 1 \text{AU}$, and 15% of the donor mass is transferred over 50 orbital periods via 500 discrete transfer events. In addition to these isolated two-body tests, we also perform proof-of-concept hierarchical three-body validations for the $q_0 = 2.0$ case to assess whether the closure remains well behaved in the presence of an external gravitational perturber.

4.1 Orbital evolution with closure

Figure 1 shows the evolution of the osculating semi-major axis with angular momentum closure active. The framework reproduces the expected behavior: expansion for $q_0 < 1$, weak expansion for $q_0 = 1$, and contraction for $q_0 > 1$.

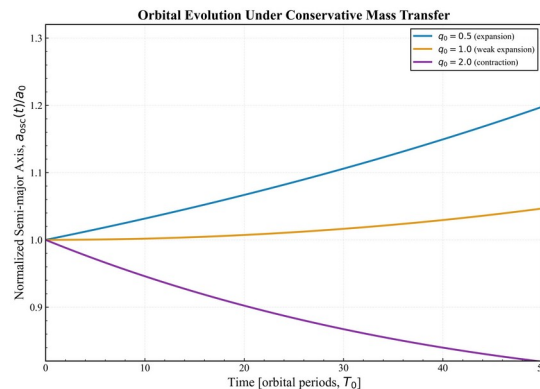


Figure 1. Orbital evolution under conservative mass transfer with angular momentum closure. The osculating semi-major axis, normalized to its initial value, is shown as a function of time for three initial mass ratios: $q_0 = 0.5$ (expansion), $q_0 = 1.0$ (weak expansion), and $q_0 = 2.0$ (contraction). The simulation transfers 15% of the donor mass over 500 events across 50 orbital periods. The orbital response is consistent with the analytical prediction from Equation (eq3): expansion when the donor is less massive, contraction when the donor is more massive.

4.2 Conservation properties

Figure 2 shows that angular momentum is conserved to $< 10^{-13}$ relative precision when

closure is active. Energy evolves consistently with orbital changes and is not artificially constrained.

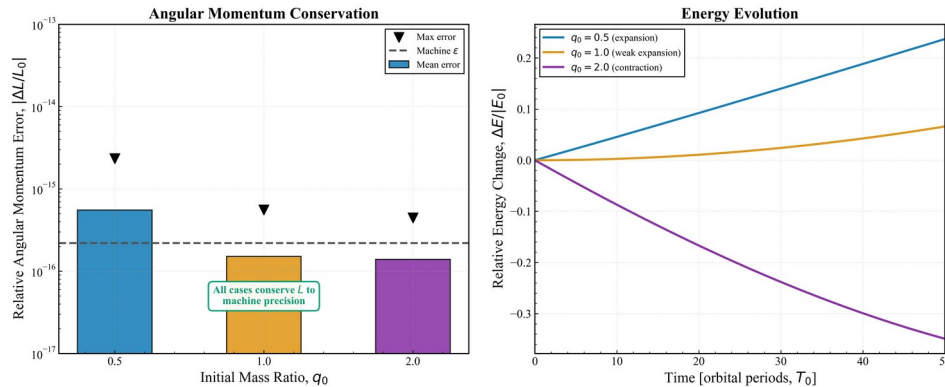


Figure 2. Conservation diagnostics with angular momentum closure active. Left panel: Relative angular momentum error for each mass ratio. Mean and maximum errors are shown; all remain below machine precision ($\epsilon \sim 2.2 \times 10^{-16}$) confirming that the closure condition enforces angular momentum conservation to numerical precision. Right panel: Relative energy change over time. Energy is not constrained by the closure and evolves consistently with the orbital changes: it increases for expanding orbits ($q_0 = 0.5$) and decreases for contracting orbits ($q_0 = 2.0$).

4.3 Failure mode without closure

To demonstrate that the mapping analyzed here requires explicit closure to reproduce the conservative limit, we run identical simulations

with the angular momentum closure (Equation 18) disabled. Mass is conserved by construction, and we enforce the COM frame ($P \approx 0$), but the closure condition is not imposed.

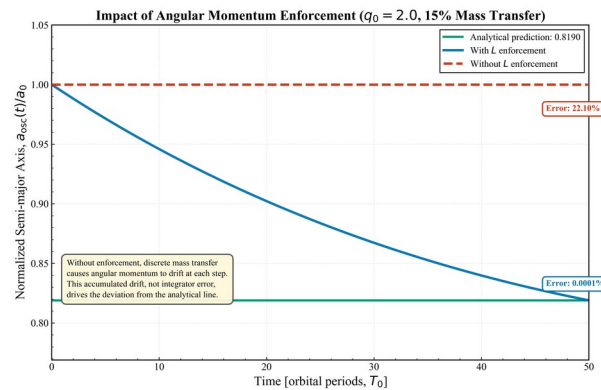


Figure 3. Impact of angular momentum enforcement on orbital evolution for $q_0 = 2.0$ with 15% mass transfer. The solid blue curve shows the simulation with the angular momentum closure condition active, achieving 0.0001% agreement with the analytical prediction (dashed line at $a_t/a_0 = 0.8190$). The dashed red curve shows the simulation without closure, where uncorrected angular momentum drift accumulates at each transfer event, producing a 22.10% systematic deviation. The drift arises from the instantaneous relocation mapping, not from integrator error.

Figure 3 compares the orbital evolution with and without closure for the $q_0=2.0$ case. Without closure, the system exhibits secular orbital drift: the semi-major axis deviates systematically from the analytical conservative prediction. As derived in Section 3.3, each instantaneous transfer event changes the system's angular momentum by an amount proportional to $(M_1^2 - M_2^2)/M_2^2$. For $q_0=2.0$, the donor is more massive, so $\Delta L > 0$ at each event, and L increases secularly. The integrator itself conserves L to $\sim 10^{-14}$ precision between transfer events; the drift is entirely due to the unclosed mapping.

Table 2 quantifies the breakdown for all three mass ratios over 500 transfer events. With

closure, the mapping reproduces the continuous conservative limit to $10^{-4}\%$ precision. Without closure, systematic deviations of 4.45–22.10% emerge. As derived in Section 3.4, the unclosed mapping produces $L_f/L_0 = (1-f)(1+f q_0)$ and leaves the orbital separation unchanged ($a_f/a_0=1$). The numerical L_f/L_0 values in Table 2 match this analytical prediction to four significant figures. The orbital deviations reported in the “Error (unclosed)” column correspond exactly to $|1 - (a_f/a_0)_{\text{cons}}| / (a_f/a_0)_{\text{cons}}$, confirming that the unclosed orbit is frozen at a_0 while the conservative prediction departs from it.

Table 2. Orbital deviation with and without angular momentum closure after 500 transfer events. The mapping is defined in the COM frame ($|P|/P_{\text{scale}} \sim 10^{-17}$ – 10^{-19}); the deviation arises from unclosed angular momentum redistribution. The L_f/L_0 values match the analytical prediction $(1-f)(1+f q_0)$ from Equation (eq11) to four significant figures.

Case	q_0	Error (closed)	Error (unclosed)	L_f/L_0 (unclosed)	$ P /P_{\text{scale}}$
A	0.5	0.0001%	16.51%	0.9137	7.72×10^{-17}
B	1.0	0.0000%	4.45%	0.9775	2.15×10^{-19}
C	2.0	0.0001%	22.10%	1.1050	3.00×10^{-17}

The residual drift in Case B ($q_0=1$) arises because the mass ratio evolves away from unity during transfer: as M_1 decreases and M_2 increases, subsequent events have $M_1 < M_2$, producing net $\Delta L < 0$.

4.4 Comparison with analytical predictions

Table 3 and Figure 4 compare numerical and analytical results for the closed case. Agreement is at the $10^{-4}\%$ level in all cases. This agreement is expected by construction when the closure is correctly implemented; it serves as validation of the implementation, not as a discovery.

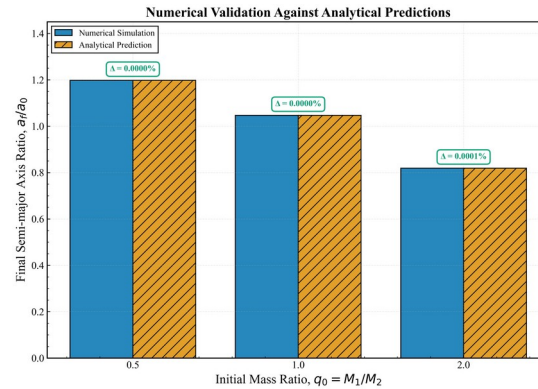


Figure 4. Numerical validation against analytical predictions for the final semi-major axis ratio after 15% conservative mass transfer. For each initial mass ratio ($q_0 = 0.5, 1.0, 2.0$), the numerical result (solid bar) is compared with the analytical prediction from Equation 3 (hatched bar). The percentage difference Δ is annotated above each pair. Agreement is at the 0.0001% level or better in all cases, confirming that the angular momentum closure condition reproduces the continuous conservative limit.

Table 3. Validation results for 15% mass transfer with angular momentum closure.

Case	q_0	$(a_f/a_0)_{num}$	$(a_f/a_0)_{ana}$	Error	Behavior
A	0.5	1.1977	1.1977	0.0001%	Expansion
B	1.0	1.0466	1.0466	0.0000%	Weak expansion
C	2.0	0.8190	0.8190	0.0001%	Contraction

4.5 Three-body proof-of-concept

To test whether the closure remains consistent outside the isolated two-body setting, we performed proof-of-concept hierarchical three-body experiments. The inner binary was initialized with the same $q_0=2.0$ configuration used in Case C, while a third body was placed on a wider circular orbit around the inner-binary center of mass. We considered two illustrative perturber configurations: a weak perturber with $m_3=0.3 M_\odot$ at $a_{out}=8.0$ AU, and a stronger perturber with $m_3=0.8 M_\odot$ at $a_{out}=5.0$ AU.

In this setting, the inner binary is not expected to follow the isolated conservative two-body solution exactly, because external torques from the third body are present. The relevant diagnostic is therefore not exact agreement with Equation 3, but whether the mass-transfer operator introduces additional artificial angular-momentum drift in the inner pair. To test this, we compare closed and unclosed transfer updates using the final inner-binary semi-major axis ratio and the mean absolute transfer-induced jump in the inner-pair orbital angular momentum per transfer event, measured relative to the instantaneous pre-transfer inner-pair angular momentum.

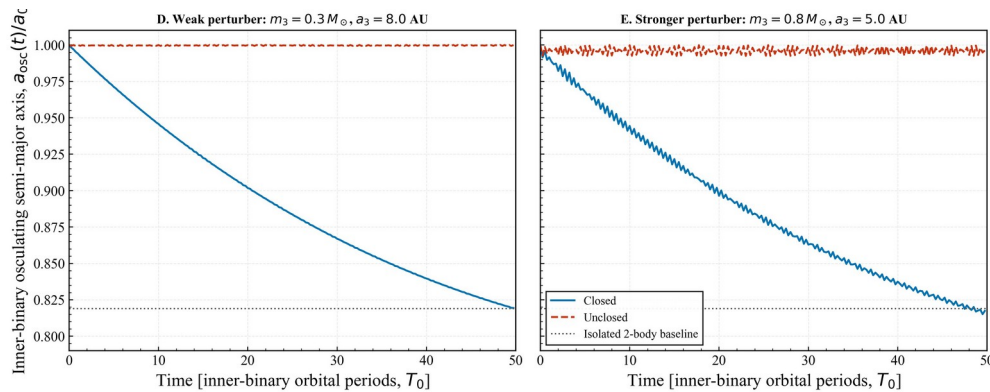


Figure 5. Three-body proof-of-concept validation of the closure condition for an inner binary with initial mass ratio $q_0 = 2.0$ undergoing 15% donor-mass transfer. Left: weak perturber case with $m_3 = 0.3 M_{\text{sun}}$ at $a_{\text{out}} = 8.0$ AU. Right: stronger perturber case with $m_3 = 0.8 M_{\text{sun}}$ at $a_{\text{out}} = 5.0$ AU. The blue solid curve shows the closed mapping, the red dashed curve shows the unclosed mapping, and the gray dotted line shows the isolated two-body conservative reference value. In both perturbed cases, the unclosed mapping leaves the inner orbit artificially near its initial separation, whereas the closed mapping preserves the expected qualitative contraction. The stronger-perturber case exhibits small oscillations in the closed curve due to external perturbations, but the overall contraction trend is maintained.

Figure 5 shows that in both perturbed cases the unclosed mapping leaves the inner orbit near its initial separation, whereas the closed mapping preserves the expected qualitative contraction of the inner binary. In the stronger-perturber case, the closed curve exhibits small oscillations due to the third body's perturbation, but the overall contracting trend is maintained. Table 4 quantifies this behavior: the unclosed mapping gives $(a_f/a_0)_{\text{unclosed}} = 0.9995$ and 0.9970 in the weak and stronger perturber cases, respectively, while the closed mapping gives $(a_f/a_0)_{\text{closed}} = 0.8187$ and 0.8157 . Most importantly, the closure suppresses the mean transfer-induced inner-pair angular-momentum jump from $\sim 2.00 \times 10^{-4}$ per event to $\sim 10^{-16}$, demonstrating that it removes the artificial operator-induced drift even in the presence of an external perturber. These tests are not a general N -body validation, but they support the relevance of the closure in representative few-body settings.

Table 4. Three-body proof-of-concept for an inner binary with $q_0 = 2.0$ undergoing 15% donor-mass transfer. The closure is applied only to the transfer event on the inner pair. Here Jump denotes the mean absolute transfer-induced change in the inner-pair orbital angular momentum per event, measured relative to the instantaneous pre-transfer inner-pair angular momentum.

Case	$m_3 (M_{\odot})$	$a_{\text{out}} (\text{AU})$	$(a_f/a_0)_{\text{closed}}$	$(a_f/a_0)_{\text{unclosed}}$	$\text{Jump}_{\text{closed}}$	$\text{Jump}_{\text{unclosed}}$
D	0.3	8.0	0.8187	0.9995	9.46×10^{-17}	2.00×10^{-4}
E	0.8	5.0	0.8157	0.9970	1.06×10^{-16}	2.00×10^{-4}

5 Discussion

Our central result is that the instantaneous relocation mapping defined in Section 3.2 does not reproduce the continuous conservative limit unless an explicit angular momentum closure is imposed. This is not a numerical subtlety but a structural distinction: the mapping does not resolve the internal dynamics that, in a real continuous transfer process, would redistribute angular momentum on dynamical timescales.

5.1 The role of linear vs. angular momentum conservation

When mass is instantaneously relocated in position-velocity space using the mapping defined in Section 3.2, enforcing total linear momentum conservation ($P=0$) does not guarantee angular momentum conservation. Equation 8 quantitatively shows how the angular momentum change depends on the difference in specific angular momenta of the two stellar components, which are generally unequal.

This distinction between linear and angular momentum conservation does not arise in the continuous analytical framework where most binary evolution theory is developed. In continuous transfer, both conservation laws are simultaneously satisfied through the underlying dynamics. In a discrete mapping, however, they are independent constraints. The mapping analyzed here satisfies one but not the other.

5.2 Physical interpretation of the closure

The closure condition (Equation 18) represents the net effect of unresolved internal dynamics—torques, tidal adjustments, and orbital response

—that occur during continuous mass transfer but are absent in the instantaneous mapping. It enforces precisely the angular momentum budget that continuous transfer would maintain through internal torques and dynamical adjustment.

5.3 Why the unclosed orbit is frozen

The result $a_f/a_0=1$ for the unclosed mapping (Equation 13) may appear surprising: one might expect mass redistribution to produce either orbital expansion or contraction. The explanation is that no dynamical force acts during the mapping step. Mass is instantaneously relocated and assigned new velocities, the model injects/removes kinetic energy but does not apply a force through displacement. The orbital energy $E=-GM_1M_2/(2a)$ changes only through the change in the product M_1M_2 , which is determined entirely by the mass update and is independent of the velocity prescription. Meanwhile, L also changes, but in a way that is exactly compensated by the change in reduced mass $\mu=M_1M_2/M_{tot}$ when forming the ratio $a \propto L^2/\mu^2$. The cancellation is exact, not approximate, and holds for arbitrary f and q_0 . The closure condition breaks this cancellation by injecting the tangential velocity adjustment that represents the missing orbital response.

5.4 Why smaller steps do not help

An expectation that may arise is that taking the limit $\delta m \rightarrow 0$ would remove the need for explicit closure. For the specific mapping defined in Section 3.2, this expectation is incorrect. The angular momentum drift per event is proportional to δm (Equation 8), but transferring

a fixed total mass requires proportionally more events as δm decreases. The cumulative drift therefore scales with total transferred mass, not with step size. Indeed, the closed-form result $L_f/L_0 = (1-f)(1+f q_0)$ (Equation 11) is independent of the number of transfer events, confirming that the drift is set by the total transferred fraction f , not by the step size δm . Formally, the limits do not commute.

$$\lim_{\delta m \rightarrow 0} (\text{unclosed instantaneous relocation}) \neq \text{continuous conservative transfer}, \quad (\text{eq 20})$$

This is an operator-splitting issue, not a convergence issue. The closure condition must be applied at every step.

The non-convergence discussed here is not a universal property of all instantaneous transfer schemes. Different velocity assignments—such as depositing mass with the donor’s velocity or with an intermediate stream velocity—would produce different per-step angular momentum changes and may exhibit different limiting behavior. Our analysis applies specifically to the accretor-comoving deposition rule defined in Section 3.2, other velocity-assignment prescriptions may exhibit different limiting

5.5 Scaling with transferred mass fraction

The orbital deviations reported in Section 4.3 arise from transferring 15% of the donor mass, a value chosen to illustrate the behavior of the mapping. The analytical result of Section 3.4 predicts that the unclosed mapping always yields $a_f/a_0 = 1$, so the orbital deviation relative to the conservative prediction is entirely determined by how far the conservative result departs from unity. Since $(a_f/a_0)_{\text{cons}} = [(1-f)(1+f q_0)]^{-2}$ (Equation 3), the deviation scales with both f and q_0 in a predictable way.

Table 5. Orbital deviation in final semi-major axis ratio (%) without closure as a function of total transferred mass fraction. Numerical values match the analytical prediction

$|1 - (a_f/a_0)_{\text{cons}}| / (a_f/a_0)_{\text{cons}}$ to within simulation precision.

f_{transfer}	$q_0 = 0.5$	$q_0 = 1.0$	$q_0 = 2.0$
1%	1.01	0.02	1.97
2%	2.03	0.08	3.88
5%	5.18	0.50	9.20
10%	10.70	1.99	16.64
15%	16.51	4.45	22.10

To verify this, we repeated the unclosed 2%, 5%, 10%, and 15%. Table 5 compares the simulations at mass-transfer fractions of 1%, numerical deviations with the analytical

prediction (Equation 16). The numerical results confirm the analytical prediction across all mass ratios and transfer fractions. For small f , the drift in L is $\sim f(q_0 - 1)$ (Equation 14), and the orbital deviation increases $\sim$ linearly with f . Even at 1–2% mass transfer, deviations of 1–4% persist for unequal-mass systems. Whether such deviations are significant depends on the application: for studies near the boundary between orbital expansion and contraction, even modest systematic shifts could affect qualitative conclusions.

5.6 Applicability

This work does not claim that existing analytic binary evolution or population synthesis codes exhibit the drift analyzed here. As discussed in Section 1, codes such as BSE (9), COSMIC (10), and MESA (8) evolve orbital elements directly using conservation-law formulas and do not employ the position-velocity mapping we analyze.

The closure condition is most directly relevant to dynamical N -body simulations in which interacting binaries are embedded in multi-body gravitational environments. In such simulations, mass transfer cannot always be handled through analytic orbital-element updates, because the binary's orbit is simultaneously perturbed by external bodies. Discrete position-velocity updates may then be necessary, and the closure condition provides a simple way to maintain angular-momentum consistency during such updates.

The proof-of-concept three-body tests in Section 4.5 provide initial support for this

interpretation. In those experiments, the closure suppresses spurious transfer-induced angular-momentum jumps in the inner binary even when an external perturber is present, under both weak and stronger perturbations. However, these tests are intentionally limited and do not constitute a general validation for arbitrary few-body or many-body systems.

5.7 Perspective: closure as a diagnostic quantity

In the framework developed here, the closure condition (Equation 18) is introduced to restore angular momentum conservation when mass transfer is represented through the instantaneous relocation mapping defined in Section 3.2. In this sense, the closure primarily functions as a consistency condition: it enforces the orbital response that continuous conservative transfer would produce but that the discrete operator does not explicitly resolve.

However, the magnitude of the closure applied at each transfer event can also be interpreted as a measure of the mismatch between the assumed instantaneous mapping and the orbital response required by the underlying dynamics. Because this mismatch depends on quantities such as the component velocities, orbital geometry, and instantaneous system configuration, it may in principle reflect properties of the dynamical environment in which the binary evolves.

In perturbed few-body or many-body systems, for example, external gravitational interactions can modify the instantaneous orbital state of a binary prior to each transfer event. If the same discrete transfer operator is applied uniformly across many systems, the closure required to

maintain angular-momentum consistency may therefore vary systematically with the strength and frequency of such perturbations. In this sense, the closure can be viewed not only as a numerical correction but also as a proxy indicator of unresolved dynamical coupling between mass transfer and orbital evolution.

This perspective suggests a possible diagnostic application at the population level. If closure magnitudes were evaluated across large ensembles of interacting binaries distributed across different galactic environments—for example, in dynamical N -body simulations or population synthesis models that employ discrete transfer updates—their statistical distribution could potentially encode information about the dynamical complexity of those environments. Regions with higher stellar densities, stronger perturbation fields, or more frequent gravitational encounters might exhibit systematically larger or more variable closure magnitudes than dynamically quieter regions. In this way, closure statistics sampled across a galactic binary population could in principle serve as a proxy signal for regional differences in stellar encounter rates, density structure, or dynamical heterogeneity—properties that are otherwise difficult to infer without detailed stream-resolved modeling.

This line of reasoning bears a methodological resemblance to cases in the history of computational physics where patterns in numerical behavior pointed toward deeper physical structure. The Fermi–Pasta–Ulam–Tsingou (FPUT) experiment is a well-known example: what appeared to be anomalous

behavior in a numerical simulation of coupled oscillators ultimately revealed hidden mathematical structure in nonlinear dynamics, eventually leading to the theory of solitons (17). The reasoning pattern—observing systematic behavior in a numerical quantity and asking whether it reflects properties of the underlying system—is conceptually similar to what is proposed here. An important disanalogy should be noted, however: in the FPUT case the anomaly arose within a physically accurate model, whereas the closure here originates from a numerical operator already known to be artificial. The prospects for discovering genuinely new physics via this route are therefore more modest. Nevertheless, the methodological point stands: if the numerical framework is fixed and applied consistently across many systems, patterns in closure behavior may still reveal relative differences in the dynamical environments of those systems, even when the absolute magnitude of the closure carries no direct physical meaning.

The idea is necessarily speculative and lies beyond the scope of the present work, which focuses on the analytical properties of the instantaneous relocation mapping itself. Nevertheless, it illustrates a broader methodological point: quantities introduced to enforce numerical consistency may also provide useful diagnostics of the physical systems being modeled when interpreted at the population level.

5.8 Limitations

This framework uses deliberate simplifications, including point masses (no Roche overflow

criterion; 13), circular orbits only, no non-conservative effects, and a single specific velocity-assignment rule. These limitations are appropriate in context, as the goal is to isolate the kinematic consequences of the mapping definition under classical assumptions. The closure condition itself would require modification for eccentric orbits, where tangential velocity rescaling also affects eccentricity.

5.9 Suggested research directions

Several natural extensions can increase the scope and depth of this research.

5.9.1 Eccentric orbits

In eccentric orbits, the angular momentum change per transfer event (Equation 8) becomes phase-dependent, and tangential velocity rescaling would also affect orbital energy and eccentricity. Extending the closure to constrain both L and E simultaneously would broaden the framework's applicability.

5.9.2 Non-conservative mass loss

Real mass-transfer episodes often involve mass loss from the system, carrying away angular momentum (14). For non-conservative transfer, the target angular momentum after each step would need to account for the angular momentum carried away by the lost mass.

5.9.3 Multi-body validation

The proof-of-concept three-body experiments in Section 4.5 show that the closure suppresses artificial transfer-induced drift in perturbed few-body settings spanning both weak and stronger hierarchical perturbations. A natural next step is

to extend this analysis to broader regions of parameter space, including non-coplanar configurations, eccentric outer orbits, non-hierarchical encounters, and higher-multiplicity systems, in order to test the robustness of the closure under more general N -body conditions.

5.9.4 Comparison with resolved-stream simulations

Smoothed Particle Hydrodynamics (SPH) and grid-based codes resolve the mass-transfer stream directly. Comparing their predictions with the instantaneous mapping (with and without closure) could offer independent verification of the physical interpretation.

6 Conclusions

We have analyzed the angular momentum properties of a specific instantaneous mass-transfer mapping in which transferred mass is deposited on the accretor with the accretor's velocity. The analysis yields three principal results.

First, the mapping changes orbital angular momentum at each transfer event by an amount proportional to $\delta m(M_1^2 - M_2^2)/M_2^2$ (Section 3.3), even when total linear momentum is exactly conserved.

Second, we derive the cumulative drift in closed form: $L_f/L_0 = (1-f)(1+f q_0)$ (Section 3.4). This formula, verified numerically to four significant figures, reveals that the unclosed mapping leaves the orbital separation exactly unchanged ($a_f/a_0=1$). The orbit is frozen because the angular momentum drift and reduced-mass change cancel precisely. All orbital deviations

reported in this paper—including the 1–22% discrepancies relative to conservative predictions and their scaling with transferred mass fraction (Section 5.5)—follow quantitatively from this single analytical result.

Third, a closure condition—uniform rescaling of tangential velocities—restores angular momentum conservation and achieves $10^{-4}\%$ agreement with analytical predictions. The closure represents the unresolved internal dynamics (torques, orbital adjustments) that occur during continuous transfer but are absent in the instantaneous mapping.

This work does not assess any specific astrophysical code or implementation. The closure condition is most directly relevant to dynamical N -body simulations where mass transfer must be implemented through discrete position-velocity updates rather than analytic orbital-element formulas. Our proof-of-concept three-body experiments show that the closure continues to suppress artificial transfer-induced

angular-momentum jumps in the inner binary even in the presence of an external perturber, for both weak and stronger hierarchical perturbations, although broader validation across general few-body and many-body configurations remains future work. Our open-source reference implementation provides a diagnostic baseline for such applications. More broadly, the results highlight the importance of explicitly defining discrete mass-transfer operators when translating continuous conservative models into particle-based simulations.

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Data availability

The source code is available at <https://github.com/fireheartjerry/binary-mass-transfer-framework>

Abbreviations

AU – Astronomical Unit (1.496×10^{11} m); COM – Center of Mass; $M_{\odot}$ – Solar Mass (1.989×10^{30} kg); MESA – Modules for Experiments in Stellar Astrophysics; SI – International System of Units; SPH – Smoothed Particle Hydrodynamics.

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