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Beyond Sudoku: New puzzle variants based on a novel mathematical mapping

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Abstract

This paper introduces a novel class of Latin squares that includes Sudoku squares as a subset. We defined a new structure by applying a ceiling-based function to each entry of a Latin square of order $n=r^2$, grouping values into blocks, and requiring that each block contained each integer from 1 to r exactly r times. This defined an intermediate grid type that bridges standard Latin squares and Sudoku grids. Furthermore, we introduced a bijective mapping ϕ on the space of Latin squares, where each row in $\phi(A)$ encoded the column positions of a given value in the original square A. This mapping enabled a new method for generating Sudoku-like grids by transforming Latin squares in a structured way. We present mathematical foundations, formal definitions, and proofs establishing the properties of this new class of grids and their relation to classical Sudoku puzzles.

Keywords

Sudoku, Latin squares, Sudoku-generating squares, Bijective mapping of Latin squares, Cyclic mapping of Latin squares, Puzzle grid

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1. Introduction

Sudoku is a globally recognized and highly popular puzzle game, known for its intriguing challenge. A Sudoku grid is a specific form of Latin square, which is a two-dimensional array consisting of integers ranging from 1 to n, where each integer appears exactly once in every row and every column. In its traditional form, Sudoku is presented as a 9×9 grid, divided into nine smaller 3×3 blocks. Each of these blocks is assumed to contain all the numbers from 1 to 9, with no repetition within the block, the row, or the column.

While many people enjoy Sudoku primarily as a stimulating brain game, the puzzle has also attracted significant academic attention. Numerous studies (1-6) have explored various aspects of Sudoku, including solution strategies, generalized puzzle forms, the enumeration of possible solutions, and connections to other mathematical fields such as Group theory and Graph theory. These investigations have enriched the understanding of Sudoku beyond its recreational appeal, revealing deeper mathematical insights.

In this paper, we offer a novel mathematical perspective on Sudoku. Specifically, we introduce a new class of Latin squares, which includes Sudoku grids as a subset, and define a bijective mapping that generates Sudoku grids from these extended Latin squares. Unlike traditional methods, our approach defines a new intermediate category of puzzle grids that lies between standard Sudoku and pure Latin squares, providing a fresh extension to the conventional Sudoku format. The mapping and the associated mathematical theory reveal new properties and offer new opportunities for both puzzle design and mathematical exploration. We present detailed proof and theoretical insights that underpin the proposed method, demonstrating the novel contributions of our work.

2. Methods

In this section, we provide the necessary mathematical definitions that will serve as the foundation for the concepts discussed throughout the remainder of the paper. Let us fix integers $r \geq 2$ and $n = r^2$. In this paper, we will use the terms arrays, squares, and matrices interchangeably to refer to square arrangements of numbers. To begin, we define the set of Latin squares (7) as follows.

2.1 *Definition 1.*

The set L_n consists of all $n \times n$ Latin squares. That is, A is an element of L_n if A is an $n \times n$ array containing integers from 1 to n, such that each number appears exactly once in each row and exactly once in each column. Before we introduce a new type of grid, we first introduce some useful index notation. For $A \in L_n$, $A_{i,j}$ denotes the entry in the i-th row and j-th column of A for $1 \leq i, j \leq n$. Additionally, we define $\widetilde{A}_{k,l}$ as the $r \times r$ block of A consisting of rows $(k-1)r+1$ through kr and columns $(l-1)r+1$ through lr, where $1 \leq k, l \leq r$. For example, $\widetilde{A}_{1,1}$ refers to the block of A containing the first r rows and the first r columns, while $\widetilde{A}_{r,r}$ refers to the block containing the last r rows and columns. We now define a function g mapping the set $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,r\}$.

2.2 *Definition 2.*

For each integer i, we define $g(i) = \lceil \frac{i}{r} \rceil$, where $\lceil x \rceil$ denotes the ceiling function of x. This means that $g(i) = k$ if $(k-1)r+1 \leq i \leq kr$ for $k = 1, \ldots, r$. Applying the function g to each entry of $A \in L_n$ results in an $n \times n$ array $g(A)$, where the (i,j)-th entry of $g(A)$ is

$g(A_{i,j})$. Now that we have established the necessary notation and function, we are ready to define a new type of grid that lies between standard Sudoku grids and Latin squares.

2.3 *Definition 3.*
We define the set G_n as a subset of L_n, where $G_n = \{ A \in L_n:$ any block $g(\widetilde{A}_{k,l}), 1 \le k, l \le r$ consists of r occurrences of each integer from 1 to r. Alternatively, if $B = g(A)$, we can replace $g(\widetilde{A}_{k,l})$ with $B_{k,l}$ in the definition. Note that if S_n denotes the set of all Sudoku squares in L_n, it follows that $S_n \subseteq G_n \subseteq L_n$. As an illustrative example, consider the following Latin square for $n = 4$

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 4 & 1 & 3 \end{bmatrix} \in L_4 \qquad \text{(eq1)}$$

If we apply the function g to A, we obtain $B = g(A)$ as follows

$$B = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Since each 2×2 block of B contains exactly two 1s and two 2s, we conclude that $A \in G_4$. However, it is evident that A is not a Sudoku square, i.e., $A \notin S_4$. Finally, we define the main function of this paper, which maps elements of L_n to new structures.

2.4 *Definition 4.*
We define a mapping ϕ on the set L_n. Given $A \in L_n$, we define $\phi(A)$ such that $\phi(A)_{k,i} = j$ if and only if $A_{i,j} = k$, for $1 \le i, j \le n$.

The definition of the mapping $\phi(A)$ is intuitively clear and can be understood by considering how the rows of the new matrix are constructed. To construct $\phi(A)$, we proceed row by row, but the key idea is that the entries of $\phi(A)$ correspond to the positions of the numbers from the original matrix A.

The first row of $\phi(A)$ is obtained by looking at the positions of the number 1 in each row of A. Specifically, for each row of A, we identify the column index where the number 1 appears. These column indices then form the entries of the first row of $\phi(A)$. Similarly, the second row of $\phi(A)$ is constructed by identifying the positions of number 2 in each row of A. This process is then repeated for each subsequent number, up to n. Through this iterative process, the rows of $\phi(A)$ are systematically formed based on the positions of the numbers in the original matrix A.

In more formal terms, if $A_{i,j} = k$, this means that the number k is located at the i-th row and j-th column of A. Therefore, for the k-th row of $\phi(A)$, the i-th entry will be the column index j where the number k appears in the i-th row of A.

For example, we compute $\phi(A)$ for the square $A \in L_4$ given in (eq1). To find the first row of $\phi(A)$, we look for the positions of the number 1 in each row of A. In the first row of A, 1 is in column 1. In the second row of A, 1 is in column 4. In the third row of A, 1 is in column 2. In the fourth row of A, 1 is in column 3. Hence, the first row of $\phi(A)$ is $(1,4,2,3)$. Similarly, applying the mapping ϕ to other rows of A, we obtain the following

$$\phi(A)=\begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{bmatrix}.$$

The mapping ϕ is chosen for its simplicity and direct applicability in the context of our work. We define ϕ in such a way that for any given entry $A_{i,j}=k$ in a Latin square A, the mapping $\phi(A)$ is defined as $\phi(A)_{k,i}=j$. This is essentially a circular rotation of the tuple (i,j,k), which preserves the structure of the Latin square while reassigning the entries in a systematic manner.

One of the key advantages of this mapping is that it generates a new Latin square $\phi(A)$, as shown in the next section. This property ensures that we are working within the well-defined space of Latin squares and preserves the combinatorial properties that are central to our study.

Another important reason for selecting ϕ is that it has a finite order, which means we do not need to apply the mapping infinitely to generate new Latin squares or to explore different properties. This finite order is particularly useful in our analysis and allows us to systematically generate and study new Sudoku-generating grids.

Furthermore, ϕ is directly tied to our definition of "Sudoku-generating squares", which are introduced in the next section. The function plays a central role in bridging the gap between Latin squares and Sudoku grids, allowing us to extend the classical Sudoku puzzle structure in a novel way.

While ϕ is the primary mapping used in our study, we can also consider an alternative mapping ψ, defined by $\psi(A)_{k,j}=i$. This alternative approach allows us to construct $\psi(A)$ column by column, as opposed to the row-by-row construction of $\phi(A)$. Both mappings, ϕ and ψ, are mathematically equivalent in that they yield the same set of results. Specifically, all the results derived using ϕ can also be obtained using ψ, and vice versa. In the following section, we will delve into several interesting and significant properties of the mapping ϕ.

3. Results

In this section, we explore several intriguing properties of the mapping ϕ, shedding light on its unique characteristics and implications. Additionally, we introduce and define a novel class of squares that occupy an intermediate position between standard Sudoku grids and Latin squares. These new structures offer a compelling hybrid between the two, capturing elements from both while exhibiting distinct properties that are of particular interest in the context of this study.

We begin by proving that the mapping ϕ is a bijection, i.e., it is both one-to-one (injective) and onto (surjective).

3.1 *Theorem 5*
$\phi: L_n \to L_n$ is bijective. We prove this as follows.

3.1.1 *Step 1.* Show that $\phi(L_n) \subseteq L_n$, i.e., $\phi(A) \in L_n$ for any $A \in L_n$. Suppose that two entries in the k-th row of the matrix $\phi(A)$ are identical. Specifically, suppose there exist indices l and l' such that $\phi(A)_{k,l} = \phi(A)_{k,l'} = j$ for some value j. Then,by the definition of ϕ, we would have $A_{l,j} = k$ and $A_{l',j} = k$, which implies $l = l'$. Similarly, suppose that two entries in the l-th column of $\phi(A)$ are identical. Then, $\phi(A)_{k,l} = \phi(A)_{k',l} = j$ for some k, k', and j. Then, we have $A_{l,j} = k$ and $A_{l,j} = k'$, which implies $k = k'$. Thus, we have shown that no number appears more than once in any row or column of $\phi(A)$, meaning that $\phi(A) \in L_n$. Therefore, $\phi(L_n) \subseteq L_n$.

3.1.2 *Step 2.* Show that ϕ is onto. Next, we demonstrate that ϕ is onto, meaning that for any element $B \in L_n$, there exists an element $A \in L_n$ such that $\phi(A) = B$. Given any $B \in L_n$, we define an $n \times n$ array A such that $A_{i, B_{k,i}} = k$ for all $1 \le i, k \le n$. This construction ensures that each row of A consists of all numbers from 1 to n, because for each i, $\{B_{k,i} : k = 1, \ldots, n\} = \{1, \ldots, n\}$. To show that A is indeed a Latin square, we must verify that each column of A also contains all numbers from 1 to n. Suppose that for a fixed j, we have $A_{i,j} = A_{i',j}$ for some $i \ne i'$. This would imply that there exist distinct k and k' such that $B_{k,i} = j$ and $B_{k',i} = j$ (since $B \in L_n$). Thus, $A_{i,j} = A_{i',j}$ implies $A_{i,B_{k,i}} = A_{i',B_{k',i}}$, which leads to the conclusion that $k = k'$ by way of constructing A from B, and hence, $j = B_{k,i} = B_{k',i} = B_{k,i'}$, which forces $i = i'$. Therefore, A has no repetitions in any column, implying that $A \in L_n$. Finally, it is straightforward to verify that $\phi(A) = B$ by the definition of ϕ. Thus, ϕ is onto.

3.1.3 *Step 3.* Show that ϕ is one-to-one. To prove that ϕ is one-to-one, suppose that $\phi(A) = \phi(A')$ for two elements A and A' in L_n. This means that for any given i, j, we have $\phi(A)_{k,i} = \phi(A')_{k,i} = j$ for some k, which implies that $A_{i,j} = A'_{i,j}(k)$. Since the entries of A and A' coincide for every row and column, we conclude that $A = A'$. Hence the mapping ϕ is injective. We now demonstrate that the mapping ϕ has order 3.

3.2 *Theorem 6*
We prove this as follows. ϕ^3 is the identity mapping. In other words, $\phi(\phi(\phi(A))) = A$ for any $A \in L_n$. Let $A \in L_n$. Define the intermediate mappings as $B = \phi(A), C = \phi(B), D = \phi(C)$. By the definition of ϕ, we have $D_{i,j} = k \Longleftrightarrow C_{j,k} = i \Longleftrightarrow B_{k,i} = j \Longleftrightarrow A_{i,j} = k$ for all i, j. Thus, $D = A$, meaning that applying ϕ three times to A results in the original array A. Since $\phi^3(A) = A$ for any $A \in L_n$, it follows that ϕ^3 is the identity mapping. An interesting and noteworthy result concerning the mapping ϕ is its preservation of the set G_n, as expressed in the following theorem.

3.3 *Theorem 7*
We prove this as follows. The mapping $\phi: L_n \to L_n$ preserves the set G_n. In other words, for any element $A \in G_n$, the image $\phi(A)$ also belongs to G_n, which can be formally written as $\phi(G_n) \subseteq G_n$. Since the map ϕ is injective and G_n is a finite set, the inclusion implies that $\phi(G_n) = G_n$. To prove this result, we begin by assuming the contrary. Suppose for some $A \in G_n$, the image of A under ϕ, denoted by $B = \phi(A)$

does not belong to G_n. In other words, $B \notin G_n$ From the definition of G_n, this assumption implies that there exist indices k, l, and s such that the number of entries in the block $g(\widetilde{A}_{k,l})$ whose values are equal to s is less than r. Formally, this condition can be written as $|\{(i,j) \in I_k \times I_l : g(B_{i,j}) = s\}| < r$, where $I_k = \{i : (k-1)r + 1 \le i \le kr\}$ and $|S|$ denotes the number of elements of a set S. This inequality can be rewritten as $|\{(i,j) \in I_k \times I_l : B_{i,j} \in I_s\}| < r$. Now, since $B_{i,j} = \phi(A_{i,j})$, we observe that $B_{i,j} = m$ implies $A_{j,m} = i$. Thus, the inequality above is equivalent to $|\{(i,j) \in I_k \times I_l : A_{j,m} = i, m \in I_s\}| < r$, which tells us that the number of entries in the block $\widetilde{A}_{l,s}$, whose images under g are in I_k, is less than r. This presents a contradiction because, by the definition of G_n, the number of such entries in $\widetilde{A}_{l,s}$ should be exactly r. Therefore, our assumption that $\phi(A) \notin G_n$ must be false. Hence, we conclude that $\phi(A) \in G_n$ for all $A \in G_n$, proving that ϕ preserves G_n.

Since $S_n \subset L_n$, Theorem 7 implies that if A is a Sudoku, then $\phi(A) \in G_n$. Equivalently, we can infer that if A is a Latin square and $\phi(A) \notin G_n$, then A is not a Sudoku. This result establishes a clear criterion for identifying Sudoku squares based on their transformation under ϕ. If the transformation does not preserve the group structure, A cannot be a Sudoku.

However, it is important to note that the converse is not true: if $\phi(A) \in G_n$, it does not necessarily imply that A is a Sudoku. In other words, while $\phi(A) \in G_n$ implies $A \in G_n$, it does not necessarily mean that A is a Sudoku. Such examples will be discussed in the next section.

Finally, we introduce the notion of Sudoku-generating squares, which are Latin squares that can be transformed into Sudoku squares through the mapping ϕ. This concept is formalized as follows.

3.4 *Definition 8*

$A \in L_n$ is called a Sudoku-generating square if A is not a Sudoku, but there exists an integer k such that $\phi^k(A)$ is a Sudoku. This definition highlights the idea that a Sudoku-generating square can eventually become a Sudoku after a finite number of applications of ϕ. In fact, since ϕ^3 is the identity mapping, we only need to consider $k=1$ or $k=2$ for the transformation to yield a Sudoku.

It is already known that $S_n \subseteq G_n$. However, we will now show that the set of Sudoku-generating squares is itself a subset of G_n, which means that every Sudoku-generating square also belongs to the group of squares that preserve the group structure.

3.5 *Theorem 9*

If $A \in L_n$ is a Sudoku-generating square, then $A \in G_n$. We prove this as follows. By the definition of a Sudoku-generating square, there exists an integer $k=1$ or $k=2$ such that $B = \phi^k(A) \in S_n$. Since ϕ^3 is the identity mapping, we can write A as $A = \phi^{3-k}(B)$. Since $B \in S_n$, it follows that $A \in G_n$. Therefore, any Sudoku-generating square is also an element of G_n, as desired.

This proof demonstrates that the set of Sudoku-generating squares is indeed a subset of the set of squares that preserve the group structure, G_n. Thus, these squares exhibit an important structural property, which further connects the concepts of Sudoku and Latin squares.

4. Discussion

Recall the relationship between the sets $S_n \subseteq G_n \subseteq L_n$. Let $S_n^{'}$ represent the subset of L_n consisting of Sudoku-generating squares. Based on this definition, we know that $S_n \cap S_n^{'} = \varnothing$, as the set $S_n^{'}$ specifically excludes Sudoku squares. Moreover, by Theorem 9, it follows that $S_n^{'} \subset G_n$. Thus, S_n and $S_n^{'}$ are disjoint subsets of G_n.In this section, we will demonstrate that for $n=4$, $S_4^{'}=G_4 \setminus S_4$, which means that the set of Sudoku-generating squares for $n=4$ is exactly the set of all squares in G_4 that are not Sudoku squares. This result is particularly significant because it establishes a clear and distinct separation between the two sets: $S_4^{'}$ and S_4. This clean separation implies that for $n=4$, the identification of Sudoku-generating squares is straightforward.

To illustrate this, consider a non-Sudoku Latin square $A \in L_4$. In this case, determining whether A is a Sudoku-generating square is a simple process. Specifically, it is sufficient to check whether $A \in G_4$, which can be done with relative ease. If we establish that $A \in G_4$, then we can immediately conclude that A must be a Sudoku-generating square. This is due to the fact that $S_4^{'}=G_4 \setminus S_4$, ensuring that every non-Sudoku square in G_4 is a Sudoku-generating square.

In contrast, for $n \geq 9$, we will show that $S_n^{'}=G_n \setminus S_n$. This implies that $G_n, n \geq 9$ contains Latin squares that are neither Sudoku squares nor Sudoku-generating squares. Consequently, for a non-Sudoku Latin square $A \in L_n$ with $n \geq 9$, simply determining whether $A \in G_n$ does not automatically imply that A is a Sudoku-generating square. To verify whether such a square A is a Sudoku-generating square, additional steps are required. Specifically, we must compute $\phi^k(A)$ for $k=1,2$ and check whether $\phi(A)$ or $\phi^2(A)$ is a Sudoku square.

4.1 *Latin squares of order 4*

When $n=4$, we have the following results. $|L_4|=576$ (the total number of Latin squares; see (8)). $|S_4|=288$ (the number of Sudoku squares; see (9)). $|S_4^{'}|=128$ (the number of Sudoku-generating squares). $|G_4|=|S_4|+|S_4^{'}|=416$ ($\{S_4, S_4^{'}\}$ is a partition of G_4). $|L_4 \setminus G_4|=160$ (the number of Latin squares that are neither Sudoku nor Sudoku-generating squares). We prove this as follows.

To begin, recall the notation used for the indices when $n=4$, specifically, $I_1=\{1,2\}$ and $I_2=\{3,4\}$. The set $S_4^{'}$ includes matrices that we will classify into two distinct formats. The first format is given by the matrix

$$A=\begin{bmatrix} a & d & b & c \\ d & a & c & b \\ c & b & a & d \\ b & c & d & a \end{bmatrix} \qquad \text{(eq2)}$$

where $\{\{a,b\},\{c,d\}\}=\{I_1, I_2\}$, meaning that the pairs $\{a,b\}$ and $\{c,d\}$ correspond to the sets I_1 and I_2 in some order. Furthermore, each 2×2 block within the matrix can have its diagonal and anti-diagonal entries swapped independently. For instance, the block in the top-left corner of the matrix, denoted as $\widetilde{A}_{1,1}$, can be written as either $\begin{bmatrix} a & d \\ d & a \end{bmatrix}$ or $\begin{bmatrix} d & a \\ a & d \end{bmatrix}$. To show that this matrix is indeed a Sudoku-generating square, we may assume $a \in I_1$. Why? If $a \in I_2$, then c must be in I_1. In that case, we can exchange the first two rows of matrix A with the last two rows, and the resulting form is the same as the one in (eq2), as shown

$$A = \begin{bmatrix} c & b & a & d \\ b & c & d & a \\ a & d & b & c \\ d & a & c & b \end{bmatrix}.$$

Exchanging the first two rows with the last two does not change the property that $\phi(A)$ is a Sudoku. Therefore, we will assume $a \in I_1$, which means that $a=1$ or $a=2$. Now, let us consider the case where $a=1$. In this case, $b=2$ and $\{c,d\}=\{3,4\}$. Matrix A takes the form

$$A = \begin{bmatrix} 1 & d & 2 & c \\ d & 1 & c & 2 \\ c & 2 & 1 & d \\ 2 & c & d & 1 \end{bmatrix}.$$

The result of applying the transformation ϕ to the matrix A depends on the specific values chosen for c and d. When $c=3$ and $d=4$, the resulting matrix $\phi(A)$ is

$$\phi(A) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}.$$

Alternatively, when $c=4$ and $d=3$, the resulting matrix $\phi(A)$ can be obtained by simply exchanging the last two rows of the previously obtained matrix. Specifically, by swapping the rows in positions 3 and 4, we obtain the following matrix

$$\phi(A) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 1 & 2 \end{bmatrix}.$$

Although the arrangement of values within the matrix changes because of the swap of c and d, both resulting matrices are valid Sudoku solutions, confirming that the matrix A is indeed a Sudoku-generating square.

Now, let us consider the case where $a=2$. If a=2, then we have b=1 and the matrix structure is similar to the previous case but with the roles of 1 and 2 swapped. It is straightforward to compute $\phi(A)$ in this scenario by simply exchanging the first two rows of the previously computed $\phi(A)$ when $a=1$. The resulting matrix $\phi(A)$ is

$$\phi(A)=\begin{bmatrix}3&4&2&1\\1&2&3&4\\4&3&1&2\\2&1&4&3\end{bmatrix}\text{ or }\begin{bmatrix}3&4&2&1\\1&2&3&4\\2&1&4&3\\4&3&1&2\end{bmatrix}.$$

In both cases, the matrix is a valid Sudoku solution. Thus, when $a=2$, the matrix A is also a Sudoku-generating square.

The number of elements of the form given in (eq2) is 64, calculated as the product of the following factors. $8=4\times2$ (4 choices for a from $\{1,2,3,4\}$ and 2 choices for d once a is selected), and $8=2\times2\times2$ (Swapping the diagonal and anti-diagonal entries in the remaining blocks $\widetilde{A}_{1,2},\widetilde{A}_{2,1}$, and $\widetilde{A}_{2,2}$ independently). The second format of Latin square that belong to $S_4^{'}$ is characterized by matrices that can be represented in one of the following distinct forms

$$A=\begin{bmatrix}a&c&b&d\\d&a&c&b\\c&b&d&a\\b&d&a&c\end{bmatrix},\begin{bmatrix}a&c&b&d\\d&a&c&b\\b&d&a&c\\c&b&d&a\end{bmatrix},\begin{bmatrix}c&a&d&b\\a&d&b&c\\d&b&c&a\\b&c&a&d\end{bmatrix},\begin{bmatrix}c&a&d&b\\a&d&b&c\\b&c&a&d\\d&b&c&a\end{bmatrix}, \qquad \text{(eq3)}$$

$$\begin{bmatrix}a&c&d&b\\d&a&b&c\\b&d&c&a\\c&b&a&d\end{bmatrix},\begin{bmatrix}a&c&d&b\\d&a&b&c\\c&b&a&d\\b&d&c&a\end{bmatrix},\begin{bmatrix}c&a&b&d\\a&d&c&b\\d&b&a&c\\b&c&d&a\end{bmatrix},\text{or}\begin{bmatrix}c&a&b&d\\a&d&c&b\\b&c&d&a\\d&b&a&c\end{bmatrix},$$

where $\{\{a,b\},\{c,d\}\}=\{I_1,I_2\}$. In this context, we will prove that any matrix of the first form in equation (eq3) is a Sudoku generating square. The following four cases will serve to demonstrate this.

4.1.1 Case: $a=1$ In this case, $b=2$ and $\{c,d\}=I_2$. The matrix A becomes

$$A=\begin{bmatrix}1&c&2&d\\d&1&c&2\\c&2&d&1\\2&d&1&c\end{bmatrix}$$

The resulting matrix $\phi(A)$ for c and d with $\{c,d\}=I_2$ is as follows

$$\phi(A)=\begin{bmatrix}1&2&4&3\\3&4&2&1\\2&3&1&4\\4&1&3&2\end{bmatrix}\text{ or }\begin{bmatrix}1&2&4&3\\3&4&2&1\\4&1&3&2\\2&3&1&4\end{bmatrix}.$$

In both cases, $\phi(A)$ is a valid Sudoku, hence A is a Sudoku-generating square.

4.1.2 Case: $b=1$
The matrix structure is similar to the previous case but with the roles of 1 and 2 swapped. It is straightforward to compute $\phi(A)$ by simply exchanging the first two rows of the previously computed $\phi(A)$ when $a=1$. Thus, A is a Sudoku-generating square.

4.1.3 Case: $c=1$
In this case, $d=2$ and $\{a,b\}=I_2$. The matrix A becomes

$$A=\begin{bmatrix} a & 1 & b & 2 \\ 2 & a & 1 & b \\ 1 & b & 2 & a \\ b & 2 & a & 1 \end{bmatrix}$$

The resulting matrix $\phi(A)$ for a and b with $\{a,b\}=I_2$ is as follows:

$$\phi(A)=\begin{bmatrix} 2 & 3 & 1 & 4 \\ 4 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 3 & 1 & 4 \\ 4 & 1 & 3 & 2 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix}.$$

As we observe, the matrix $\phi(A)$ when $c=1$ is obtained by permuting the rows of $\phi(A)$ that we previously obtained when $a=1$. This transformation does not affect the underlying structure of the matrix. Specifically, the property of $\phi(A)$ being a Sudoku matrix remains unchanged in both cases. In other words, while the arrangement of the rows may differ between the two scenarios, the matrix still satisfies the conditions of a Sudoku puzzle.

4.1.4 Case: $d=1$
The matrix structure is similar to the previous case but with the roles of 1 and 2 swapped. It is straightforward to compute $\phi(A)$ by simply exchanging the first two rows of the previously computed $\phi(A)$ when $c=1$. Thus, A is a Sudoku-generating square.

A similar analysis shows that the other seven forms in (eq3) are also Sudoku-generating squares. For the first form in (eq3), the number of possible matrices is $4\times 2=8$ (since a can be chosen from the set $\{1,2,3,4\}$ and once a is selected, the values of c and d must be chosen from the remaining elements, subject to the constraint that $\{\{a,b\},\{c,d\}\}=\{I_1,I_2\}$).

Similarly, for each of the other six forms, there are 8 different matrices. Thus, the total number of elements in this format is 64. Thus, the set S_4' contains 128 distinct Latin squares, derived from the combination of 64 Latin squares of the form presented in (eq2) and another 64 of the form in (eq3). This count, however, does not necessarily imply that S_4' contains precisely these 128 Latin squares. In other words, while we have established that there are at least 128 Latin squares in S_4', this result does not rule out the possibility of the set containing additional Latin squares beyond this number. Consequently, we can confidently conclude that $\left|S_4^{'}\right| \geq 128$. The elements of $L_4 \setminus G_4$ exhibit patterns similar to those in $S_4^{'}$. Specifically, the set $L_4 \setminus G_4$ includes matrices in two distinct formats. The first format is derived by swapping the entries b and d in (eq2) as follows

$$A = \begin{bmatrix} a & b & d & c \\ b & a & c & d \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \tag{eq4}$$

where $\{\{a,b\},\{c,d\}\} = \{I_1, I_2\}$. The second format is obtained by swapping b and c in (eq3) as follows

$$A = \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ b & c & d & a \\ c & d & a & b \end{bmatrix}, \begin{bmatrix} a & b & c & d \\ d & a & b & b \\ c & d & a & b \\ b & c & d & a \end{bmatrix}, \begin{bmatrix} b & a & d & c \\ a & d & c & b \\ d & c & b & a \\ c & b & a & d \end{bmatrix}, \begin{bmatrix} b & a & d & c \\ a & d & c & b \\ c & b & a & d \\ d & c & b & a \end{bmatrix}, \tag{eq5}$$

$$\begin{bmatrix} a & b & d & c \\ d & a & c & b \\ c & d & b & a \\ b & c & a & d \end{bmatrix}, \begin{bmatrix} a & b & d & c \\ d & a & c & b \\ b & c & a & d \\ c & d & b & a \end{bmatrix}, \begin{bmatrix} b & a & c & d \\ a & d & b & c \\ d & c & a & b \\ c & b & d & a \end{bmatrix}, \begin{bmatrix} b & a & c & d \\ a & d & b & c \\ c & b & d & a \\ d & c & a & b \end{bmatrix},$$

where $\{a,b,c,d\} = \{1,2,3,4\}$, a and c are placed in different sets of I_1 and I_2, and b and d also belong to different sets within I_1 and I_2. It is straightforward to show that while both matrix formats are Latin squares, they do not satisfy the conditions required to belong to G_4.

The number of Latin squares that correspond to the first format is 32. This is calculated as follows. There are 2 choices for the elements in the top-left corner (either from I_1 or I_2). There are 2^4 ways to swap diagonal and anti-diagonal entries within each of four 2×2 blocks. Therefore, the total number of elements in the first format is $2 \times 2^4 = 32$.

The number of elements corresponding to each matrix in (eq5) is 16. The calculation for this is as follows. 4×2 (4 choices for a from $\{1,2,3,4\}$ and 2 choices for c once a is selected). 2 (2 choices for b and d for each selection of a and c)

Therefore, the total number of elements in the second format is calculated as $16 \times 8 = 128$ When we combine the elements from both formats, we arrive at the inequality $\left|L_4 \setminus G_4\right| \geq 128 + 32 = 160$. It is important to note that, as with the case of $S_4^{'}$, this argument demonstrates that the set $L_4 \setminus G_4$ contains at least 160 distinct Latin squares, each of which is derived from either (eq4) or (eq5). However, this does not necessarily imply that $L_4 \setminus G_4$ consists exclusively of these 160 Latin squares. In other words, while we can confirm that there are at least 160 Latin squares in this set, the possibility remains that there are additional squares within $L_4 \setminus G_4$.

Consequently, the inequality $|L_4 \setminus G_4| \geq 160$ leads to the upper bound for $|G_4|$, which is given by $|G_4| = |L_4| - |L_4 \setminus G_4| \leq 576 - 160 = 416$.
Furthermore, since S_4 and $S_4^{'}$ are disjoint subsets of G_4, it follows that $|G_4| \geq |S_4| + |S_4{}'| \geq 288 + 128 = 416$. Therefore, for both inequalities to hold simultaneously, two equalities must be true. As a result, we can conclude that S_4 and $S_4^{'}$ forms a partition of G_4. Additionally, it follows that $S_4^{'}$ consists solely of the squares described in (eq2) and (eq3), while $L_4 \setminus G_4$ is composed exclusively of the squares given in (eq4) and (eq5).

We know that there are 160 Latin squares that are neither Sudoku squares nor Sudoku-generating squares. The following example illustrates such a case

$$A = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix}.$$

To verify that A is not a Sudoku-generating square, we inspect the matrix $g(A)$

$$g(A) = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}.$$

Upon inspection, it is clear that the 2×2 blocks in $g(A)$ do not exhibit an equal distribution of 1s and 2s. This failure confirms that $A \notin G_4$. Since $G_4 = S_4 \cup S_4^{'}$ and A does not belong to G_4, it follows that A is not a Sudoku-generating square.

To summarize, Latin squares $A \in L_4$ can be classified into three categories. *Sudoku squares:* These are Latin squares that are already valid Sudoku squares. There are 288 such Latin squares. *Sudoku-generating squares:* These are Latin squares that are not Sudoku squares themselves, but when transformed using the function ϕ, they generate a Sudoku square. There are 128 such Latin squares. *Other Latin squares:* These are Latin squares where neither A nor $\phi(A)$ is a Sudoku square. Furthermore, $\phi^2(A)$ (or any subsequent applications of ϕ) also does not generate a Sudoku square. There are 160 such Latin squares.

4.2 *Latin squares of order* $n \geq 9$

The number of Latin squares grows exponentially as n increases, making it impractical to list the sizes of L_n and its subsets. Instead of focusing on these large numbers, we will present examples that highlight the differences between $S_9^{'}$ and $G_9 \setminus S_9$. These examples will demonstrate the more complex relationships that arise for $n \geq 9$, providing further insight into the properties of Sudoku and Sudoku-generating squares in the context of larger Latin squares.

Consider the following example of a Latin square

$$A = \begin{bmatrix} 1 & 2 & 6 & 7 & 8 & 3 & 4 & 5 & 9 \\ 2 & 6 & 7 & 8 & 3 & 4 & 5 & 9 & 1 \\ 6 & 7 & 8 & 3 & 4 & 5 & 9 & 1 & 2 \\ 7 & 8 & 3 & 4 & 5 & 9 & 1 & 2 & 6 \\ 8 & 3 & 4 & 5 & 9 & 1 & 2 & 6 & 7 \\ 3 & 4 & 5 & 9 & 1 & 2 & 6 & 7 & 8 \\ 4 & 5 & 9 & 1 & 2 & 6 & 7 & 8 & 3 \\ 5 & 9 & 1 & 2 & 6 & 7 & 8 & 3 & 4 \\ 9 & 1 & 2 & 6 & 7 & 8 & 3 & 4 & 5 \end{bmatrix}.$$

Each row of A is generated by circularly shifting the previous row by one position to the left, which guarantees that $A \in L_9$. Moreover, since every 3×3 block of $g(A)$ as shown below contains exactly three 1s, three 2s, and three 3s, we can conclude that $A \in G_9$

$$g(A) = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \\ 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 & 1 \\ 3 & 3 & 1 & 2 & 2 & 3 & 1 & 1 & 2 \\ 3 & 1 & 2 & 2 & 3 & 1 & 1 & 2 & 3 \\ 1 & 2 & 2 & 3 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & 1 & 1 & 2 & 3 & 3 & 1 \\ 2 & 3 & 1 & 1 & 2 & 3 & 3 & 1 & 2 \\ 3 & 1 & 1 & 2 & 3 & 3 & 1 & 2 & 2 \end{bmatrix}.$$

It is clear that $A \notin S_9$. To explore further, we compute $\phi(A)$ and $\phi^2(A)$

$$\phi(A) = \begin{bmatrix} 1 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 9 & 8 & 7 & 6 & 5 & 4 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1 & 9 & 8 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 9 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 9 \\ 3 & 2 & 1 & 9 & 8 & 7 & 6 & 5 & 4 \\ 4 & 3 & 2 & 1 & 9 & 8 & 7 & 6 & 5 \\ 5 & 4 & 3 & 2 & 1 & 9 & 8 & 7 & 6 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

and

$$\phi^2(A)=\begin{bmatrix} 1 & 2 & 6 & 7 & 8 & 3 & 4 & 5 & 9 \\ 9 & 1 & 5 & 6 & 7 & 2 & 3 & 4 & 8 \\ 8 & 9 & 4 & 5 & 6 & 1 & 2 & 3 & 7 \\ 7 & 8 & 3 & 4 & 5 & 9 & 1 & 2 & 6 \\ 6 & 7 & 2 & 3 & 4 & 8 & 9 & 1 & 5 \\ 5 & 6 & 1 & 2 & 3 & 7 & 8 & 9 & 4 \\ 4 & 5 & 9 & 1 & 2 & 6 & 7 & 8 & 3 \\ 3 & 4 & 8 & 9 & 1 & 5 & 6 & 7 & 2 \\ 2 & 3 & 7 & 8 & 9 & 4 & 5 & 6 & 1 \end{bmatrix}.$$

Upon inspecting the first 3×3 blocks of both $\phi(A)$ and $\phi^2(A)$, we observe that they contain duplicated numbers, meaning that neither of these matrices satisfies the Sudoku condition. As a result, these transformations show that $A\in G_9$ but $A\notin S_9$ and $A\notin S_9^{'}$. This example clearly demonstrates that the relation $S_9^{'}=G_9\setminus S_9$ does not hold. Similar examples —specifically, Latin squares where each row is obtained by circularly shifting the previous row by one position to the left—can be utilized to demonstrate that $S_n^{'}\neq G_n\setminus S_n$ for $n\geq 9$. Specifically, there exists an element $A\in G_n$ that is neither a Sudoku square nor a Sudoku-generating square.

6. Conclusion

In this paper, we introduced an intriguing one-to-one mapping ϕ from the set of Latin squares L_n onto itself. By restricting the domain to G_n, we demonstrated that the mapping remains one-to-one but now maps G_n onto itself. One of the key properties of this mapping is that it has order 3, meaning that applying the mapping three times returns the identity, leaving the original Latin square unchanged.

Using this mapping, we further defined Sudoku-generating Latin squares. These squares lie in an interesting position between Sudoku squares and general Latin squares, acting as a bridge between these two categories. While Sudoku puzzles are known for their challenging nature, solving a Sudoku-generating square seems to be an even more complex and intriguing puzzle. This extension of Sudoku provides an interesting variant that not only offers a greater challenge but also opens new avenues for exploration in the context of combinatorial games and puzzle theory.

The primary focus of this paper is to introduce the new concept of Sudoku-generating squares and to explore some of their interesting properties. However, the paper does not provide systematic strategies for identifying such squares. Developing methods for systematically finding Sudoku-generating squares could be a promising and impactful area for future research.

7. References

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